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ON A SELF-SIMILAR SOLUTION OF A TWO-DIMENSIONAL FILTRATION PROBLEM IN REGIONS WITH MOVING BOUNDARIES

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We determine a family of self-similar solutions of a two-dimensional problem involving the filtration of an incompressible liquid in regions with moving boundaries. Our work is based on a method developed by Galin for solving the problem of settling of water cones in a gravitational field [1 – 3]. Following this method, we reduce the problem to one of finding an analytic function of a complex variable and the time, which effects a conformal mapping of the filtration region onto a strip and satisfies a special nonlinear condition on the boundary. For the solution of a problem of this kind Galin proposed the method of successive approximations.

1. Statement of the problem. We consider filtration of an incompressible liquid in a region bounded by two infinite contours, Γ_1 and Γ_2 (see Fig. 1a), one of which we assume to be fixed and the other moving. We denote the moving contour by $\Gamma_2(t)$. We assume the pressure constant on both contours, $p = p_0$ on Γ_1 and $p = 0$ on Γ_2. This corresponds to the case when liquid is pumped into the stratum along contour Γ_1 which is the boundary between the liquid and gas. The quantity $p_0 = p_{\Gamma_1} - p_{\Gamma_2}$ represents then the pressure drop with the pressure in the gas region being constant. The complex potential of such motion is of the form

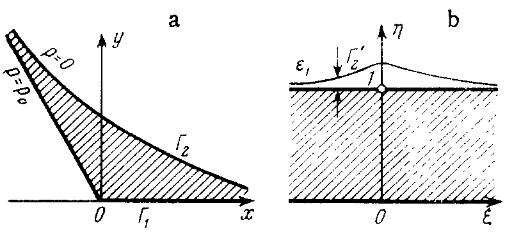

Fig. 1

$$W(z, t) = -kW_1(z, t)$$

where k is the coefficient of filtration. Moreover, $p(x, y, t) = \operatorname{Re} W_1(z, t)$. Let the function $z = z(t, \zeta)$ map the plane of the complex variable $z = x + iy$ conformally onto the strip $0 \leqslant \eta \leqslant 1$ in the plane $\zeta = \xi + i\eta$ in such a way that the contour Γ_1 goes over into the line $\eta = 0$ and the contour Γ_2 goes over into the line $\eta = 1$ (Fig. 1b). In addition, we require that $z(t, 0) = 0$. The following conditions must then be satisfied on the boundary of this strip:

$$\zeta = \xi + i: \ \operatorname{Re}[W_1(\zeta, t)] = 0$$
$$\zeta = \xi + i0: \ \operatorname{Re}[W_1(\zeta, t)] = p_0$$

The solution of this problem in the ζ plane is obviously

$$W_1(\zeta, t) = ip_0\zeta + p_0 \tag{1.1}$$

It remains then to find the mapping function $z = z(\zeta, t)$. To obtain the conditions determining this function we use Galin's method [1 – 3]. Let the contour Γ_2 be displaced normally by an amount $\varepsilon(\Gamma_2, t)$ in the time interval Δt. Then

$$\varepsilon = \frac{v_n}{m}\Delta t = -\frac{k}{m}\frac{\partial p}{\partial n}\Delta t$$

Here m is the porosity. Since it is clear that (see Eq. (1.1))

$$\frac{\partial p}{\partial n} = \left|\frac{\partial W_1}{\partial \zeta}\right|\left|\frac{\partial \zeta}{\partial z}\right| = p_0\left|\frac{\partial \zeta}{\partial z}\right| = \frac{p_0}{|\partial z / \partial \zeta|}$$

we obtain the following expression for the amount of the displacement:

$$\varepsilon = \frac{kp_0}{|\partial z / \partial \zeta|}\frac{\Delta t}{m}$$

In using the mapping $z(t, \zeta)$ corresponding to time t, we note that the new position of the contour Γ_2' in the ζ plane at the instant $(t + \Delta t)$ will differ from the line $\eta = i$ by the amount of the normal displacement ε_1. It is obvious that

$$\varepsilon_1 = \frac{\varepsilon}{|\partial z / \partial \zeta|} = \frac{kp_0}{|\partial z / \partial \zeta|^2}\frac{\Delta t}{m} \qquad (1.2)$$

Let the function $\zeta_1(\zeta)$ map the filtration region bounded by the contours Γ'_1 and Γ_2' onto the strip $0 \leqslant \eta \leqslant 1$. Then the magnitude of the difference appearing within the brackets in the expression

$$\zeta_1(\zeta) = \zeta + [\zeta_1(\zeta) - \zeta]$$

will be small, and it is obvious that

$$\operatorname{Im}[\zeta_1(\zeta) - \zeta]_{\eta=1} = \varepsilon_1, \qquad \operatorname{Im}[\zeta_1(\zeta) - \zeta]_{\eta=0} = 0 \qquad (1.3)$$

In addition $z(t + \Delta t, \zeta) = z(t, \zeta_1(\zeta))$. Using the last equations, we can write

$$z(t + \Delta t, \zeta) - z(t, \zeta) = \frac{\partial z}{\partial t}\Delta t + \ldots$$

$$z(t + \Delta t, \zeta) - z(t, \zeta) = z(t, \zeta_1) - z(t, \zeta) = \frac{\partial z}{\partial \zeta}[\zeta_1 - \zeta] + \ldots$$

where the dots denote infinitesimals of higher order. Then

$$\frac{\partial z / \partial t}{\partial z / \partial \zeta} = \frac{\zeta_1 - \zeta}{\Delta t} + \ldots$$

Using Eqs. (1.2) and (1.3), we obtain

$$\operatorname{Im}\frac{\partial z / \partial t}{\partial z / \partial \zeta}\bigg|_{\eta=1} = \frac{kp_0}{m|\partial z / \partial \zeta|^2}, \qquad \operatorname{Im}\frac{\partial z / \partial t}{\partial z / \partial \zeta}\bigg|_{\eta=0} = 0 \qquad (1.4)$$

These expressions constitute the nonlinear boundary conditions for determining the mapping function $z(t, \zeta)$. After transforming these conditions, we can rewrite them in the following equivalent form:

$$\operatorname{Im}\frac{\partial z}{\partial \zeta}\frac{\partial \bar{z}}{\partial \tau} = -1 \quad \text{for } \zeta = \xi + i \qquad (1.5)$$

$$\operatorname{Im}\frac{\partial z}{\partial \zeta}\frac{\partial \bar{z}}{\partial \tau} = 0 \quad \text{for } \zeta = \xi + i0$$

$$(\tau = p_0 kt / m, \ \bar{z} = x - iy)$$

2. Self-similar solution. We seek the mapping function $z(t, \zeta)$ for our problem in the form

$$z(\tau, \zeta) = \sqrt{\tau}\, z^*(\zeta)$$

Here $z^*(\zeta)$ is an analytic function of the complex variable $\zeta = \xi + i\eta$, defined in the strip $0 \leqslant \eta \leqslant 1$. On the boundaries of the strip, as a consequence of the relations (1.5), the following conditions must be satisfied:

$$\operatorname{Im}\frac{dz^*}{d\zeta}\bar{z}^* = -2 \quad \text{for} \quad \zeta = \xi + i \tag{2.1}$$

$$\operatorname{Im}\frac{dz^*}{d\zeta}\bar{z}^* = 0 \quad \text{for} \quad \zeta = \xi + i0$$

In addition, we require that $z^*(0) = 0$.

The question of finding a complete solution of this problem remains open, however, we can point out a certain class of its solutions. Let us seek those solutions for which the quantity

$$\operatorname{Im}\left(\frac{dz^*}{d\zeta}\bar{z}^*\right)$$

is a function of the single variable η. This condition leads to the conclusion that the quantity $|z^*|^2$ must be representable in the form

$$|z^*|^2 = \alpha(\xi) + \beta(\eta)$$

Since $z^*(\zeta)$ is an analytic function of ζ, it follows that $\alpha(\xi)$ and $\beta(\eta)$ must be connected by the differential equation

$$\alpha''_{\xi\xi} + \beta''_{\eta\eta} = \frac{\alpha'^2_{\xi} + \beta'^2_{\eta}}{\alpha + \beta}$$

Without giving the details of the transformations of this equation, we merely remark that all of its solutions can be obtained from the system of equations

$$\left(\frac{d\alpha}{d\xi}\right)^2 = c_1 + c_2\alpha + c_3\alpha^2, \qquad \left(\frac{d\beta}{d\eta}\right)^2 = -c_1 + c_2\beta - c_3\beta^2$$

where c_1, c_2 and c_3 are arbitrary constants. From this system we determine the following essentially distinct types of solutions satisfying the boundary conditions (2.1):

$$1)\ z^* = \sqrt{2}\,\zeta, \qquad z^* = \sqrt{\frac{2p_0kt}{m}}\,\zeta$$

$$2)\ z^* = \frac{2}{\sqrt{\lambda\sin 2\lambda}}\operatorname{sh}\lambda\zeta, \qquad z^* = \sqrt{\frac{4p_0kt}{m\lambda\sin 2\lambda}}\operatorname{sh}\lambda\zeta$$

The first of these solutions corresponds to a one-dimensional motion of the liquid with streamlines parallel to the OY-axis; the second of these is essentially multi-dimensional in nature. The flow picture in the x, y plane is depicted in Fig. 2. The equipotential curves of the resulting self-similar solution are given by the moving hyperbolas ($\eta = \text{const}$, $0 < \eta \leqslant 1$)

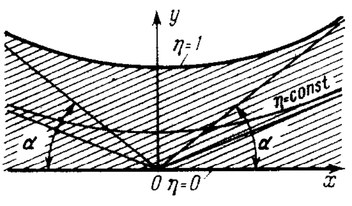

Fig. 2

$$\left(\frac{y}{\sin\lambda\eta}\right)^2 - \left(\frac{x}{\cos\lambda\eta}\right)^2 = \frac{4p_0kt}{m\lambda\sin 2\lambda}$$

while the streamlines are the ellipses ($\xi = \text{const}$, $0 < \xi < \infty$)

$$\left(\frac{x}{\operatorname{sh}\lambda\xi}\right)^2 + \left(\frac{y}{\operatorname{ch}\lambda\xi}\right)^2 = \frac{4p_0kt}{m\lambda\sin 2\lambda}$$

Initially, the liquid occupies the two sectors adjacent to the OX-axis, the sector angle being given by $\alpha = \operatorname{arctg}\lambda$, $0 < \lambda < \infty$.

In conclusion, we note that, in spite of its artificial nature, the solution we have found may prove to be useful for the solution of certain special filtration problems; it may also be used for determining the accuracy of approximate solutions and computational algorithms.

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SEPARATION OF THE ELASTICITY THEORY EQUATIONS WITH RADIAL INHOMOGENEITY

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The separation of a system of three elasticity theory equations in the static case to a system of two equations and one independent equation for a space with a radial inhomogeneity is presented in a spherical coordinate system. These equations are solved by separation of variables for specific kinds of radial inhomogeneity. In particular, solutions are found for the Lamé coefficients $\mu = \text{const}$, $\lambda(r)$ is an arbitrary function, $\mu = \mu_0 r^\beta$, $\lambda = \lambda_0 r^\beta$.

While methods of solving problems associated with the equilibrium of an elastic homogeneous sphere have been studied sufficiently [1], problems with spherical symmetry of the boundary conditions have mainly been solved for an inhomogeneous sphere [2, 3].

For a particular kind of inhomogeneity dependent on one Cartesian coordinate, the equations have been separated completely in [4]. A system of three equations with a radial inhomogeneity in a spherical coordinate system is separated below by a method analogous to [4].

1. The equilibrium equations in displacements with a radial inhomogeneity and no mass forces are

$$(\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{u} + \mathbf{i}_r\lambda'\,\mathrm{div}\,\mathbf{u} + \mu\left(\mathbf{i}_r \times \mathrm{rot}\,\mathbf{u} + 2\frac{\partial u}{\partial r}\right) = 0 \qquad (1.1)$$

Here $\lambda(r)$ and $\mu(r)$ are the Lamé coefficients dependent on the radius, $\mathbf{i}_r$ is the unit vector in the radial direction, and u is the displacement vector. Let us write (1.1) in matrix form in spherical coordinates

$$\| a_{ik} \| \,\mathrm{col}\,(u_r, u_\theta, u_\varphi) = 0 \qquad (1.2)$$

$$a_{11} = \mu\,[D_\theta^\circ D_\theta + D_\varphi^2] + \frac{\partial}{\partial r}\left[\lambda D^\circ + 2\mu\frac{\partial}{\partial r}\right] + \frac{4\mu}{r}\left[\frac{\partial}{\partial r} - \frac{1}{r}\right]$$